

# On the Zero-Point Energy of a Conducting Spherical Shell

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## Abstract

The zero-point energy of a conducting spherical shell is evaluated by imposing boundary conditions on the potential  $A_\mu$ , and on the ghost fields. The scheme requires that temporal and tangential components of  $A_\mu$  perturbations should vanish at the boundary, jointly with the gauge-averaging functional, first chosen of the Lorenz type. Gauge invariance of such boundary conditions is then obtained provided that the ghost fields vanish at the boundary. Normal and longitudinal modes of the potential obey an entangled system of eigenvalue equations, whose solution is a linear combination of Bessel functions under

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the above assumptions, and with the help of the Feynman choice for a dimensionless gauge parameter. Interestingly, ghost modes cancel exactly the contribution to the Casimir energy resulting from transverse and temporal modes of  $A_\mu$ , jointly with the decoupled normal mode of  $A_\mu$ . Moreover, normal and longitudinal components of  $A_\mu$  for the interior and the exterior problem give a result in complete agreement with the one first found by Boyer, who studied instead boundary conditions involving TE and TM modes of the electromagnetic field. The coupled eigenvalue equations for perturbative modes of the potential are also analyzed in the axial gauge, and for arbitrary values of the gauge parameter. The set of modes which contribute to the Casimir energy is then drastically changed, and comparison with the case of a flat boundary sheds some light on the key features of the Casimir energy in non-covariant gauges.

## 1. INTRODUCTION

One of the most striking applications of quantum field theory to physical processes consists of the analysis of zero-point energies.<sup>1–7</sup> Indeed, it is by now well known that zero-point energies of quantized fields are infinite, whereas suitable differences in zero-point energies are finite and measurable. A good example is provided by the quantum theory of the electromagnetic field. In that case, the energy of two perfectly conducting plates of area  $A$  and separation  $d$  is found to be<sup>1</sup>

$$\Delta E = -\frac{\pi^2 \hbar c A}{720 d^3}. \quad (1.1)$$

One thus has an attraction between such plates. Further to this effect, the zero-point energy of a conducting spherical shell was calculated in Ref. 2 for the first time, giving, however, a different sign with respect to the case of parallel plates:

$$\Delta E(R) \cong 0.09 \frac{\hbar c}{2R}, \quad (1.2)$$

where  $R$  is the radius of the sphere. The result (1.2) has been confirmed, independently, by all authors working in this research field. A powerful tool turns out to be the method relying on the use of the generalized  $\zeta$ -function. The boundary conditions of the problem (see below) are of the Dirichlet or Robin type. The corresponding Laplace-type operators, say  $Q$ , are then self-adjoint, and the analytic continuation of their  $\zeta$ -function:

$$\zeta_Q(s) \equiv \text{Tr}_{L^2}(Q^{-s}) \quad (1.3)$$

may be used to obtain the value of  $\Delta E(R)$ .<sup>6,8</sup> We here omit all details concerning  $\zeta$ -function calculations, and we focus instead on the problem of boundary conditions.

Let  $r, \theta, \varphi$  be the spherical coordinates, which are appropriate for the analysis of a conducting spherical shell. The boundary conditions require that the tangential components of the electric field should vanish, i.e.,

$$[E_\theta]_{\partial M} = 0, \quad (1.4)$$

$$[E_\varphi]_{\partial M} = 0. \quad (1.5)$$

These boundary conditions are gauge-invariant, since they involve some components of the electromagnetic-field tensor:

$$F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (1.6)$$

which is gauge-invariant whenever the connection  $\nabla$  is torsion-free, as we always assume. The modes of the electromagnetic field are then split into transverse electric (TE) or magnetic multipole:<sup>9</sup>

$$\vec{r} \cdot \vec{E} = E_r = 0, \quad (1.7)$$

and transverse magnetic (TM) or electric multipole:<sup>9</sup>

$$\vec{r} \cdot \vec{B} = B_r = 0. \quad (1.8)$$

On using the notation in Ref. 9 for spherical Bessel functions, and the formulae therein for  $E_\theta$  and  $E_\varphi$ , one finds that, in the TE case, Eqs. (1.4) and (1.5) lead to

$$[A_l j_l(kr) + B_l n_l(kr)]_{\partial M} = 0, \quad (1.9)$$

while in the TM case the vanishing of  $E_\theta$  and  $E_\varphi$  at the boundary implies that

$$\left[ \frac{d}{dr} (r(C_l j_l(kr) + D_l n_l(kr))) \right]_{\partial M} = 0. \quad (1.10)$$

Of course, if the background includes the point  $r = 0$ , the coefficients  $B_l$  and  $D_l$  should be set to zero for all values of  $l$  to obtain a regular solution. Such a singularity is instead avoided if one studies the annular region in between two concentric spheres.

The boundary conditions in the form (1.9) and (1.10) are sufficient to implement the algorithm of the  $\zeta$ -function. However, they do not tell us anything about the contribution of the ghost field. Indeed, if one follows a path-integral approach to the quantization of Maxwell theory, it is well-known that the second-order operator acting on  $A^\mu$  perturbations when the Lorenz gauge-averaging functional is chosen turns out to be, in a flat background,<sup>7</sup>

$$P_{\mu\nu} = -g_{\mu\nu}\square + \left(1 - \frac{1}{\alpha}\right) \nabla_\mu \nabla_\nu. \quad (1.11)$$

With a standard notation,  $g$  is the background metric,  $\alpha$  is a dimensionless parameter,  $\square$  is the D'Alembert operator

$$\square = -\frac{\partial^2}{\partial t^2} + \Delta,$$

where  $\Delta$  is the Laplace operator (in our problem,  $\Delta$  is considered on a disk). This split of the  $\square$  operator leads eventually to the eigenvalue equations for the Laplace operator acting on the temporal, normal and tangential components of the potential, after using the equations (2.1)–(2.3) of Sec. 2. At this stage, the potential, with its gauge transformations

$${}^\varepsilon A_\mu \equiv A_\mu + \nabla_\mu \varepsilon, \quad (1.12)$$

is viewed as a more fundamental object. Some care, however, is then necessary to ensure gauge invariance of the whole set of boundary conditions. For example, if one imposes the boundary conditions

$$[A_t]_{\partial M} = 0, \quad (1.13)$$

$$[A_\theta]_{\partial M} = 0, \quad (1.14)$$

$$[A_\varphi]_{\partial M} = 0, \quad (1.15)$$

this is enough to ensure that the boundary conditions (1.4) and (1.5) hold. However, within this framework, one has to impose yet another condition. By virtue of Eq. (1.12), the desired boundary condition involves the gauge function:

$$[\varepsilon]_{\partial M} = 0. \quad (1.16)$$

Equation (1.16) ensures that the boundary conditions (1.13)–(1.15) are preserved under the gauge transformations (1.12). What happens is that  $\varepsilon$  is expanded in harmonics on the two-sphere according to the relation

$$\varepsilon(t, r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \varepsilon_l(r) Y_{lm}(\theta, \varphi) e^{i\omega t}. \quad (1.17)$$

After a gauge transformation, one finds from (1.12)

$$[{}^\varepsilon A_t]_{\partial M} - [A_t]_{\partial M} = [\nabla_t \varepsilon]_{\partial M}, \quad (1.18)$$

$$[{}^\varepsilon A_\theta]_{\partial M} - [A_\theta]_{\partial M} = [\nabla_\theta \varepsilon]_{\partial M}, \quad (1.19)$$

$$[{}^\varepsilon A_\varphi]_{\partial M} - [A_\varphi]_{\partial M} = [\nabla_\varphi \varepsilon]_{\partial M}, \quad (1.20)$$

and by virtue of (1.17) one has

$$[\nabla_t \varepsilon]_{\partial M} = i\omega [\varepsilon]_{\partial M}, \quad (1.21)$$

$$[\nabla_\theta \varepsilon]_{\partial M} = \left[ \sum_{l=0}^{\infty} \sum_{m=-l}^l \varepsilon_l(r) Y_{lm,\theta}(\theta, \varphi) e^{i\omega t} \right]_{\partial M}, \quad (1.22)$$

$$[\nabla_\varphi \varepsilon]_{\partial M} = \left[ \sum_{l=0}^{\infty} \sum_{m=-l}^l \varepsilon_l(r) Y_{lm,\varphi}(\theta, \varphi) e^{i\omega t} \right]_{\partial M}. \quad (1.23)$$

Thus, if  $\varepsilon_l(r)$  is set to zero at the boundary  $\forall l$ , the right-hand sides of (1.21)–(1.23) vanish at  $\partial M$ . But

$$[\varepsilon_l(r)]_{\partial M} = 0 \quad \forall l$$

is precisely the condition which ensures the vanishing of  $\varepsilon$  at  $\partial M$ , and the proof of our statement is completed.

At this stage, the only boundary condition on  $A_r$  whose preservation under the transformation (1.12) is again guaranteed by Eq. (1.16) is the vanishing of the gauge-averaging functional, say  $\Phi(A)$ , at the boundary:<sup>7</sup>

$$[\Phi(A)]_{\partial M} = 0. \quad (1.24)$$

On choosing the Lorenz term  $\Phi_L(A) \equiv \nabla^b A_b$ , Eq. (1.24) leads to

$$\left[ \frac{\partial}{\partial r} (r^2 A_r) \right]_{\partial M} = 0. \quad (1.25)$$

It is only upon considering the joint effect of Eqs. (1.13)–(1.16), (1.24) and (1.25) that the whole set of boundary conditions becomes gauge-invariant. This scheme is also BRST-invariant (cf. Ref. 10). More precisely, Eq. (1.16) should be replaced, in the quantum theory, by a boundary condition on both the real and the imaginary part of a complex-valued ghost field, say<sup>7</sup>

$$[\omega]_{\partial M} = [\psi]_{\partial M} = 0. \quad (1.26)$$

Even this terminology is a bit loose, since ghost fields are really independent of each other, as stressed by DeWitt.<sup>11</sup> In the calculations of the following sections, it will be enough to consider a *real-valued gauge function* obeying Eq. (1.16) at the boundary, and then multiply the resulting contribution to the zero-point energy by  $-2$ , bearing in mind Eq. (1.26) and the fermionic nature of ghost fields. A similar analysis of boundary conditions for the Casimir effect can be found in Refs. 12 and 13. Note also that our boundary conditions on the potential lead to the fulfillment of all “perfect conductor” boundary conditions studied in Ref. 2, which involve tangential components of the electric field and the normal component of the magnetic field.

The plan of our paper is as follows. Section 2 derives the form of basis functions for the electromagnetic potential. Section 3 presents the eigenvalue conditions for the perturbation modes occurring in the Casimir calculation in the presence of spherical symmetry, and describes how the Casimir energy of a conducting spherical shell can be obtained from such eigenvalue conditions. The eigenvalue equations in the axial gauge are studied in detail in Sec. 4. Results and open problems are described in Sec. 5, relevant details are given in the Appendices.

## 2. BASIS FUNCTIONS FOR THE ELECTROMAGNETIC POTENTIAL

We know from Sec. 1 that, in our problem, the electromagnetic potential has a temporal component  $A_t$ , a normal component  $A_r$ , and tangential components, say  $A_k$  (hereafter,  $k$  refers to  $\theta$  and  $\varphi$ ). They can all be expanded in harmonics on the two-sphere, according to the standard formulas

$$A_t(t, r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_l(r) Y_{lm}(\theta, \varphi) e^{i\omega t}, \quad (2.1)$$

$$A_r(t, r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l b_l(r) Y_{lm}(\theta, \varphi) e^{i\omega t}, \quad (2.2)$$

$$A_k(t, r, \theta, \varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^l [c_l(r) \partial_k Y_{lm}(\theta, \varphi) + d_l(r) \varepsilon_{kp} \partial^p Y_{lm}(\theta, \varphi)] e^{i\omega t}. \quad (2.3)$$

This means that we are performing a Fourier analysis of the components of the electromagnetic potential. The two terms in square brackets of Eq. (2.3) refer to longitudinal and transverse modes, respectively. On setting  $\alpha = 1$  in Eq. (1.11), the evaluation of basis functions for electromagnetic perturbations can be performed after studying the action of the operator  $-g_{\mu\nu} \square$  on the components (2.1)–(2.3). For this purpose, we perform the analytic continuation  $\omega \rightarrow iM$ , which makes it possible to express the basis functions in terms of modified Bessel functions (of course, one could work equally well with  $\omega$ , which leads instead to ordinary Bessel functions). This can be shown by using the formulas (A2)–(A4) of Appendix A when  $d = 2$ . For the temporal component, one deals with the Laplacian acting on a scalar field on the three-dimensional disk:

$${}^{(3)} \triangle A_t = \frac{\partial^2 A_t}{\partial r^2} + \frac{2}{r} \frac{\partial A_t}{\partial r} + \frac{1}{r^2} {}^{(2)} \triangle A_t, \quad (2.4)$$

which leads to the eigenvalue equation

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] a_l = M^2 a_l. \quad (2.5)$$

Hereafter, we omit for simplicity any subscript for  $M^2$ , and the notation for the modes will make it sufficiently clear which spectrum is studied. The solution of Eq. (2.5) which is regular at  $r = 0$  is thus found to be

$$a_l(r) = \frac{1}{\sqrt{r}} I_{l+1/2}(Mr), \quad (2.6)$$

up to an unessential multiplicative constant.

The action of  ${}^{(3)}\Delta$  on the component  $A_r$  normal to the two-sphere is (from Eq. (A3))

$$({}^{(3)}\Delta A)_r = \frac{\partial^2 A_r}{\partial r^2} + \frac{2}{r} \frac{\partial A_r}{\partial r} + \frac{1}{r^2} ({}^{(2)}\Delta A)_r - \frac{2}{r^2} A_r - \frac{2}{r^3} A_p |^p, \quad (2.7)$$

where the stroke  $|$  denotes two-dimensional covariant differentiation on a two-sphere of unit radius. Last, the Laplacian on tangential components takes the form (cf. Eq. (A4))

$$({}^{(3)}\Delta A)_k = \frac{\partial^2 A_k}{\partial r^2} + \frac{1}{r^2} ({}^{(2)}\Delta A)_k - \frac{1}{r^2} A_k + \frac{2}{r} \partial_k A_r. \quad (2.8)$$

By virtue of the expansions (2.2) and (2.3), jointly with the properties (A5) and (A6) of spherical harmonics, Eqs. (2.7) and (2.8) lead to the eigenvalue equation

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right] d_l = M^2 d_l \quad (2.9)$$

for transverse modes, jointly with entangled eigenvalue equations for normal and longitudinal modes (here  $l \geq 1$ ):

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{(l(l+1)+2)}{r^2} \right] b_l + \frac{2l(l+1)}{r^3} c_l = M^2 b_l, \quad (2.10)$$

$$\left[ \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} \right] c_l + \frac{2}{r} b_l = M^2 c_l. \quad (2.11)$$

The mode  $b_0(r)$  is instead decoupled, and is proportional to  $\frac{I_{3/2}(Mr)}{\sqrt{r}}$  in the interior problem.

It is indeed well known that gauge modes of the Maxwell field obey a coupled set of eigenvalue equations. In arbitrary gauges, one cannot decouple these modes.<sup>14</sup> This can be proved by trying to put in diagonal form the  $2 \times 2$  operator matrix acting on the modes  $b_l$  and  $c_l$  (cf. Refs. 7 and 15). In our problem, however, with our choice of gauge-averaging functional and

gauge parameter, gauge modes can be disentangled, and a simpler method to achieve this exists.<sup>16</sup> For this purpose, we point out that, since the background is flat, if gauge modes can be decoupled, they can only reduce to linear combinations of Bessel functions, i.e.,

$$b_l(r) = \frac{B_\nu(Mr)}{\sqrt{r}}, \quad (2.12)$$

and

$$c_l(r) = C(\nu)B_\nu(Mr)\sqrt{r}. \quad (2.13)$$

With our notation,  $C(\nu)$  is some constant depending on  $\nu$ , which is obtained in turn from  $l$ . To find  $\nu$  and  $C(\nu)$ , we insert the ansatz (2.12) and (2.13) into the system of equations (2.10) and (2.11), and we require that the resulting equations should be of Bessel type for  $B_\nu(Mr)$ , i.e.,

$$B_\nu'' + \frac{B_\nu'}{Mr} - \frac{\nu^2}{M^2 r^2} B_\nu - M^2 B_\nu = 0. \quad (2.14)$$

This leads to two algebraic equations for  $\nu^2$ :

$$\nu^2 = \frac{9}{4} + l(l+1) - 2l(l+1)C, \quad (2.15)$$

and

$$\nu^2 = \frac{1}{4} + l(l+1) - \frac{2}{C}. \quad (2.16)$$

By comparison, one thus finds a second-order algebraic equation for  $C$ :

$$l(l+1)C^2 - C - 1 = 0, \quad (2.17)$$

whose roots are

$$C_+ = \frac{1}{l}, \quad (2.18)$$

and

$$C_- = -\frac{1}{(l+1)}. \quad (2.19)$$

The corresponding values of  $\nu$  are

$$\nu_+ = l - \frac{1}{2}, \quad (2.20)$$

and

$$\nu_- = l + \frac{3}{2}. \quad (2.21)$$

Hence one finds the basis functions for normal and longitudinal perturbations in the interior problem in the form

$$b_l(r) = \alpha_{1,l} \frac{I_{l+3/2}(Mr)}{\sqrt{r}} + \alpha_{2,l} \frac{I_{l-1/2}(Mr)}{\sqrt{r}}, \quad (2.22)$$

$$c_l(r) = -\frac{\alpha_{1,l}}{(l+1)} I_{l+3/2}(Mr) \sqrt{r} + \frac{\alpha_{2,l}}{l} I_{l-1/2}(Mr) \sqrt{r}, \quad (2.23)$$

whereas, from Eq. (2.9), transverse modes read

$$d_l(r) = I_{l+1/2}(Mr) \sqrt{r}. \quad (2.24)$$

Last, but not least, ghost modes obey an eigenvalue equation analogous to (2.5) (see comments after (1.26)), i.e.

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] \varepsilon_l = M^2 \varepsilon_l, \quad (2.25)$$

and hence they read

$$\varepsilon_l(r) = \frac{1}{\sqrt{r}} I_{l+1/2}(Mr). \quad (2.26)$$

In the exterior problem, i.e., for  $r$  greater than the two-sphere radius  $R$ , one has simply to replace the modified Bessel functions of first kind in Eqs. (2.6), (2.22)–(2.24) and (2.26) by modified Bessel functions of second kind, to ensure regularity at infinity.

### 3. EIGENVALUE CONDITIONS AND CASIMIR ENERGY

In our problem, ghost modes are of course decoupled from the modes for the electromagnetic potential which occur in the expansions (2.1)–(2.3). Nevertheless, this does not mean that they do not play a role in the Casimir energy calculation. By contrast, we already know from Sec. 1 (see comments after (1.16)) that boundary conditions on the ghost are strictly necessary to ensure gauge invariance of the boundary conditions on the potential. It is then clear that such boundary conditions, combined with the differential equation (2.25), lead to a ghost spectrum whose contribution to the Casimir energy can only be obtained after a detailed calculation (e.g. Green-function approach, or  $\zeta$ -function regularization). As is pointed out in Ref. 12, it is not surprising that ghost terms are important, since one has already included effects of other degrees of freedom which should be compensated for. This issue is further clarified by the analysis of the original Casimir problem: two perfectly conducting parallel plates. In a covariant formalism, one has to consider the energy-momentum tensor for ghosts, which is found to give a non-vanishing contribution to the average energy density in vacuum (see Sec. 4.5 of Ref. 7). After taking into account boundary conditions for the potential entirely analogous to our Eqs. (1.13)–(1.15) and (1.24), one then finds a renormalized value of the zero-point energy in complete agreement with the result first found by Casimir.<sup>1</sup>

By virtue of Eq. (1.13), the modes  $a_l(r)$  obey homogeneous Dirichlet conditions:

$$[a_l(r)]_{\partial M} = 0, \forall l \geq 0. \quad (3.1)$$

Moreover, Eq. (1.25) implies that the modes  $b_l$  obey the boundary conditions

$$\left[ \frac{\partial}{\partial r} r^2 b_l(r) \right]_{\partial M} = 0, \forall l \geq 0. \quad (3.2)$$

Last, the modes  $c_l$  and  $d_l$ , being the tangential modes, obey Dirichlet boundary conditions (cf. Eqs. (1.14) and (1.15))

$$[c_l(r)]_{\partial M} = [d_l(r)]_{\partial M} = 0, \forall l \geq 1. \quad (3.3)$$

On taking into account how the modes are expressed in terms of Bessel functions (see Eqs. (2.6), (2.22)–(2.24)), one thus finds five sets of eigenvalue conditions for the interior and exterior problems, respectively:

(i) Temporal modes:

$$I_{l+1/2}(MR) = 0, \forall l \geq 0, \quad (3.4a)$$

$$K_{l+1/2}(MR) = 0, \forall l \geq 0. \quad (3.4b)$$

(ii) Decoupled normal mode:

$$\left[ \frac{\partial}{\partial r} r^{3/2} I_{3/2}(Mr) \right]_{r=R} = 0, \quad (3.5a)$$

$$\left[ \frac{\partial}{\partial r} r^{3/2} K_{3/2}(Mr) \right]_{r=R} = 0. \quad (3.5b)$$

(iii) Coupled longitudinal and normal modes (here  $\nu \equiv l + 3/2$ ):

$$\begin{aligned} & (\nu - 1/2) I'_\nu(MR) I_{\nu-2}(MR) + 3(\nu - 1) \frac{I_\nu(MR)}{MR} I_{\nu-2}(MR) \\ & + (\nu - 3/2) I'_{\nu-2}(MR) I_\nu(MR) = 0, \end{aligned} \quad (3.6a)$$

$$\begin{aligned} & (\nu - 1/2) K'_\nu(MR) K_{\nu-2}(MR) + 3(\nu - 1) \frac{K_\nu(MR)}{MR} K_{\nu-2}(MR) \\ & + (\nu - 3/2) K'_{\nu-2}(MR) K_\nu(MR) = 0. \end{aligned} \quad (3.6b)$$

(iv) Transverse modes:

$$I_{l+1/2}(MR) = 0, \forall l \geq 1, \quad (3.7a)$$

$$K_{l+1/2}(MR) = 0, \forall l \geq 1. \quad (3.7b)$$

Note that, the boundary being a two-sphere, our “transverse photons” are two-dimensional, unlike other physical problems where transverse photons are three-dimensional since the boundary is a three-sphere (or another three-surface).

(v) Ghost modes (multiplying their  $\zeta$ -function by -2):

$$I_{l+1/2}(MR) = 0, \forall l \geq 0, \quad (3.8a)$$

$$K_{l+1/2}(MR) = 0, \forall l \geq 0. \quad (3.8b)$$

The eigenvalue conditions (3.5a) and (3.5b) can be re-expressed in the form

$$I_{1/2}(MR) = 0, \quad (3.9)$$

$$K_{1/2}(MR) = 0. \quad (3.10)$$

It is thus clear that, by construction, the contribution of Eqs. (3.9) and (3.10) to the Casimir energy of a conducting spherical shell cancels exactly the joint effect of Eqs. (3.4a), (3.4b), (3.7a), (3.7b), (3.8a) and (3.8b), bearing in mind the fermionic nature of ghost fields. In general, each set of boundary conditions involving a set of positive eigenvalues, say  $\{\lambda_k\}$ , contributes to the Casimir energy in a way which is clarified by the  $\zeta$ -function method, because the regularized ground-state energy is defined by the equation (for  $\text{Re}(s) > s_0 = 2$ )

$$E_0(s) \equiv -\frac{1}{2} \sum_k (\lambda_k)^{\frac{1}{2}-s} \mu^{2s} = -\frac{1}{2} \zeta\left(s - \frac{1}{2}\right) \mu^{2s}, \quad (3.11)$$

which is later analytically continued to the value  $s = 0$  in the complex- $s$  plane. Here  $\mu$  is the usual mass parameter and  $\zeta$  is the  $\zeta$ -function of the positive-definite elliptic operator  $\mathcal{B}$  with discrete spectrum  $\{\lambda_k\}$ :

$$\zeta_{\mathcal{B}}(s) \equiv \text{Tr}_{L^2}(\mathcal{B}^{-s}) = \sum_k (\lambda_k)^{-s}. \quad (3.12)$$

In other words, the regularized ground-state energy is equal to  $-\frac{1}{2}\zeta_{\mathcal{B}}(-1/2)$ . Although the eigenvalues are known only implicitly, the form of the function occurring in the mode-by-mode expression of the boundary conditions leads eventually to  $E_0(0)$ .<sup>6,8</sup>

The non-trivial part of the analysis is represented by Eqs. (3.6a) and (3.6b). These are obtained by imposing the Robin boundary conditions for normal modes, and Dirichlet

conditions for longitudinal modes. To find nontrivial solutions of the resulting linear systems of two equations in the unknowns  $\alpha_{1,l}$  and  $\alpha_{2,l}$ , the determinants of the matrices of coefficients should vanish. This leads to Eqs. (3.6a) and (3.6b). At this stage, it is more convenient to re-express such equations in terms of Bessel functions of order  $l + 1/2$ . On using the standard recurrence relations among Bessel functions and their first derivatives, one thus finds the following equivalent forms of eigenvalue conditions:

$$I_{l+1/2}(MR) \left[ I'_{l+1/2}(MR) + \frac{1}{2MR} I_{l+1/2}(MR) \right] = 0, \forall l \geq 1, \quad (3.13)$$

$$K_{l+1/2}(MR) \left[ K'_{l+1/2}(MR) + \frac{1}{2MR} K_{l+1/2}(MR) \right] = 0, \forall l \geq 1. \quad (3.14)$$

Thus, the contribution of the coupled normal and longitudinal modes splits into the sum of contributions of two scalar fields obeying Dirichlet and Robin boundary conditions, respectively, with the  $l = 0$  mode omitted. This corresponds exactly to the contributions of TE and TM modes (Eqs. (1.9) and (1.10)), and gives the same contribution as the one found by Boyer.<sup>2</sup>

#### 4. AXIAL GAUGE

As is well known, on using the Faddeev-Popov formalism, one performs Gaussian averages over gauge functionals, say  $\chi^\mu$ , by adding to the original Lagrangian a gauge-averaging term  $\chi^\mu \beta_{\mu\nu} \chi^\nu$ , where  $\beta_{\mu\nu}$  is any constant invertible matrix.<sup>11</sup> This has the effect of turning the original operator on field perturbations into a new operator which has the advantage of being non-degenerate. It can also become of Laplace type for suitable choices of some gauge parameters. Moreover, the result of a one-loop calculation is expected to be  $\chi$ - and  $\beta$ -independent, although no rigorous proof exists on manifolds with boundary.<sup>7</sup> In particular, for Maxwell theory,  $\beta_{\mu\nu}$  reduces to a  $1 \times 1$  matrix, i.e. a real-valued parameter, and  $\chi^\mu$  reduces to the familiar covariant or noncovariant gauges for Maxwell theory, e.g. Lorenz, Coulomb, axial, ... . It is therefore quite important to understand the key features of

the Casimir-energy calculations in all such gauges. For this purpose, we consider a gauge-averaging functional of the axial type, i.e.

$$\Phi(A) \equiv n^\mu A_\mu, \quad (4.1)$$

where  $n^\mu$  is the unit normal vector field  $n^\mu = (0, 1, 0, 0)$ . The resulting gauge-field operator is found to be, in our flat background,

$$P^{\mu\nu} = -g^{\mu\nu}\square + \nabla^\mu\nabla^\nu + \frac{1}{\alpha}n^\mu n^\nu. \quad (4.2)$$

Note that, unlike the case of Lorenz gauge, the  $\alpha$  parameter is dimensionful and has dimension [length]<sup>2</sup> (see Ref. 17).

Now we impose again the boundary condition according to which the gauge-averaging functional should vanish at  $\partial M$ :

$$[\Phi(A)]_{\partial M} = [n^\mu A_\mu]_{\partial M} = [A_r]_{\partial M} = 0, \quad (4.3)$$

which implies that all  $b_l$  modes vanish at the boundary (cf. Eq. (3.2)).

A further consequence of the axial gauge may be derived by acting on the field equations

$$P^{\mu\nu}A_\nu = 0 \quad (4.4)$$

with the operations of covariant differentiation and contraction with the unit normal, i.e.

$$\nabla_\mu P^{\mu\nu}A_\nu = 0, \quad (4.5)$$

$$n_\mu P^{\mu\nu}A_\nu = 0. \quad (4.6)$$

Eq. (4.5) leads, in flat space, to the differential equation

$$\frac{\partial A_r}{\partial r} + \frac{2}{r}A_r = 0. \quad (4.7)$$

This first-order equation leads to  $b_l$  modes having the form

$$b_l = \frac{b_{0,l}}{r^2}. \quad (4.8)$$

Thus, by virtue of the boundary condition (4.3), the modes  $b_l$  vanish everywhere, and hence  $A_r$  vanishes identically if the axial gauge-averaging functional is chosen with such boundary conditions.

Moreover, Eq. (4.6) leads to the equation

$$-i\omega \frac{\partial A_t}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} A_k^{[k]} = 0, \quad (4.9)$$

which implies, upon making the analytic continuation  $\omega \rightarrow iM$ ,

$$M \frac{da_l}{dr} - \frac{l(l+1)}{r^2} \frac{dc_l}{dr} = 0. \quad (4.10)$$

At this stage, the transverse modes  $d_l$  obey again the eigenvalue equation (2.9), whereas the remaining set of modes obey differential equations which, from Eq. (4.4), are found to be

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] a_l - M \frac{l(l+1)}{r^2} c_l = 0, \quad (4.11)$$

$$\frac{d^2 c_l}{dr^2} - Ma_l - M^2 c_l = 0, \quad (4.12)$$

$$a_0 = 0. \quad (4.13)$$

In particular, Eq. (4.13) is obtained from Eq. (4.12) (when  $l = 0$ ), which is a reduced form of the Eq.  $P_{k\nu} A^\nu = 0$  upon bearing in mind that  $b_l$  modes vanish everywhere.

Last, but not least, one should consider the ghost operator, which, in the axial gauge, is found to be

$$Q = -\frac{\partial}{\partial r}. \quad (4.14)$$

This leads to ghost modes having the form

$$\varepsilon_l = \varepsilon_{0,l} e^{-Mr}. \quad (4.15)$$

On the other hand, following the arguments developed in Sec. 1, one can prove that, also in the axial gauge, the ghost field should vanish at the boundary, to ensure gauge invariance

of the whole set of boundary conditions on the potential. It is then clear, from Eq. (4.15), that ghost modes vanish everywhere in the axial gauge.

On studying the system (4.10)–(4.12) one has first to prove that these three equations are compatible. This is indeed the case, because differentiation with respect to  $r$  of Eq. (4.10) leads to a second-order equation which, upon expressing  $\frac{dc_l}{dr}$  from Eq. (4.10) and  $\frac{d^2c_l}{dr^2}$  from Eq. (4.12), is found to coincide with Eq. (4.11). Thus, we have a system of two second-order differential equations for two functions  $a_l$  and  $c_l$ . However, these functions are not independent, in that they are connected by Eq. (4.10). Hence for every value of  $l$  one has one degree of freedom instead of two. Finally, we have for every  $l$  two degrees of freedom, one resulting from Eqs. (4.10)–(4.12), and another, i.e. the transverse mode  $d_l$ . Thus, an estimate of the number of degrees of freedom giving contribution to Casimir energy coincides with that in other gauges. Moreover, the parameter  $\alpha$  does not affect the Casimir energy, since  $\alpha$  does not occur in any of the eigenvalue equations.

Unfortunately, we cannot obtain the exact form of the solutions of Eqs. (4.11) and (4.12) in terms of special functions (e.g. Bessel or hypergeometric). This crucial point can be made precise by remarking that, if it were possible to disentangle the system (4.11) and (4.12), one could find some functions  $\alpha_l, \beta_l, V_l, W_l$ , say, such that the  $2 \times 2$  matrix

$$\begin{pmatrix} 1 & V_l \\ W_l & 1 \end{pmatrix} \begin{pmatrix} \hat{A}_l & \hat{B}_l \\ \hat{C}_l & \hat{D}_l \end{pmatrix} \begin{pmatrix} 1 & \alpha_l \\ \beta_l & 1 \end{pmatrix}$$

has no non-vanishing off-diagonal elements, where the operators  $\hat{A}_l, \hat{B}_l, \hat{C}_l, \hat{D}_l$  are the ones occurring in Eqs. (4.11) and (4.12) (cf. Refs. 7,14). For example, the first off-diagonal element of such matrix is the operator

$$\hat{A}_l\alpha_l + \hat{B}_l + V_l(\hat{C}_l\alpha_l + \hat{D}_l).$$

On setting to zero the coefficients of  $\frac{d^2}{dr^2}$  and  $\frac{d}{dr}$  one finds that  $\alpha_l = -V_l = \frac{\alpha_{0,l}}{r}$ , where  $\alpha_{0,l}$  is a constant. But it is then impossible to set to zero the “potential” term of this operator, i.e. its purely multiplicative part.

Nevertheless, there is some evidence that the axial gauge may be consistently used to evaluate the Casimir energy. For this purpose we find it helpful to consider a simpler

problem, i.e. the Casimir energy in the axial gauge for the case of flat boundary. In this case the basis functions are plane waves

$$A_\mu = A_{0,\mu} e^{i(k_x x + k_y y + k_z z - \omega t)}, \quad (4.16)$$

where the admissible values of  $k_x, k_y, k_z$  are determined by the boundary conditions, which are here taken to be analogous to the case of a curved boundary. Let us choose the conducting boundaries parallel to the  $x$ - and  $y$ -axes, while the vector  $n^\mu$  is directed along the  $z$ -axis. Then the condition of compatibility of field equations in the axial gauge is reduced to

$$D \equiv \det \begin{pmatrix} -k^2 & \omega k_x & \omega k_y & \omega k_z \\ \omega k_x & k_y^2 + k_z^2 - \omega^2 & -k_x k_y & -k_x k_z \\ \omega k_y & -k_x k_y & k_x^2 + k_z^2 - \omega^2 & -k_y k_z \\ \omega k_z & -k_x k_z & -k_y k_z & k_x^2 + k_y^2 - \omega^2 + \frac{1}{\alpha} \end{pmatrix} = 0. \quad (4.17)$$

Direct calculation shows that this determinant is equal to

$$D = -\frac{1}{\alpha} k_z^2 (\omega^2 - k^2)^2. \quad (4.18)$$

Hence we have reproduced the correct dispersion relation between energy  $\omega$  and wave number  $k$  (to every admissible value of  $k$  there correspond two contributions to the Casimir energy of the form  $\omega = |\vec{k}|$ ). Of course, no non-vanishing ghost modes exist, once that the axial type gauge-averaging functional is set to zero at the boundary (cf. (4.3), (4.14) and (4.15)). Interestingly, on imposing the boundary conditions, one can set to zero the  $A_z$  component of the electromagnetic potential as was done with  $A_r$  in the spherical case. In this case the compatibility condition of field equations is reduced to the vanishing of the determinant of a  $3 \times 3$  matrix, obtained by omitting the fourth row and the fourth column in Eq. (4.17). One can easily see that the determinant of this  $3 \times 3$  matrix coincides with that of the  $4 \times 4$  matrix up to a multiplicative factor  $\frac{1}{\alpha}$ , which does not affect the dispersion relation.

To sum up, a complete correspondence can be established between the key features of the axial gauge in the cases of flat and curved boundary: the  $\alpha$  parameter does not affect the Casimir energy, the component of the potential orthogonal to the boundary vanishes,

ghost modes vanish, and only two independent degrees of freedom contribute. On going from the flat to the curved case, however, the analysis of the dispersion relation is replaced by the problem of finding explicit solutions of Eqs. (4.10)–(4.12), with the corresponding eigenvalue conditions. This last technical problem goes beyond the present capabilities of the authors, but the exact results and complete correspondences established so far seem to add evidence in favour of a complete solution being in sight in the near future.

## 5. CONCLUDING REMARKS

Scientific knowledge makes progress not only when an entirely new result is first obtained, but also when alternative derivations of such properties become available, possibly showing that further important problems are in sight. The contributions of our paper fall within the latter framework. For this purpose, we have studied an approach to the evaluation of the zero-point energy of a conducting spherical shell which relies on a careful investigation of the potential and of ghost fields, with the corresponding set of boundary conditions on  $A_\mu$  perturbations and ghost modes. When Boyer first developed his calculation,<sup>2</sup> the formalism of ghost fields for the quantization of gauge fields and gravitation had just been developed,<sup>18–20</sup> and hence it is quite natural that, in the first series of papers on Casimir energies, the ghost contribution was not considered, since the emphasis was always put on TE and TM modes for the electromagnetic field. On the other hand, the Casimir energy is a peculiar property of the quantum theory, and an approach via path-integral quantization regards the potential and the ghost as more fundamental.<sup>7,11</sup> This is indeed necessary to appreciate that Maxwell theory is a gauge theory. Some of these issues were studied in Refs. 7,12,13,21,22, including calculations with ghosts in covariant gauges, but, to our knowledge, an explicit mode-by-mode analysis of the  $A_\mu$  and ghost contributions in problems with spherical symmetry was still lacking in the literature. The contributions of our investigation are as follows.

First, the basis functions for the electromagnetic potential have been found explicitly

when the Lorenz gauge-averaging functional is used. The temporal, normal, longitudinal and transverse modes have been shown to be linear combinations of spherical Bessel functions. Second, it has been proved that transverse modes of the potential are, by themselves, unable to reproduce the correct value for the Casimir energy of a conducting spherical shell. Third, it is exactly the effect of coupled longitudinal and normal modes of  $A_\mu$  which is responsible for the value of  $\Delta E$  given in Eq. (1.2), following Boyer<sup>2</sup> (see also Refs. 3,4,6). This adds evidence in favour of physical degrees of freedom for gauge theories being a concept crucially depending on the particular problem under consideration and on the boundary conditions. Further study is necessary to understand this property.<sup>14,21</sup> Fourth, ghost modes play a nontrivial role as well, in that they cancel the contribution resulting from transverse, decoupled and temporal modes of the potential. Fifth, the axial gauge-averaging functional has been used to study the Casimir energy for Boyer's problem. Unlike the case of the Lorenz gauge, ghost modes and normal modes are found to vanish, and one easily proves that the result is independent of the  $\alpha$  parameter. A complete comparison with the case of flat boundary has also been performed, getting insight into the problems of independent degrees of freedom and non-vanishing contributions to the Casimir energy.

Indeed, recent investigations of Euclidean Maxwell theory in quantum cosmological backgrounds had already shown that longitudinal, normal and ghost modes are all essential to obtain the value of the conformal anomaly and of the one-loop effective action.<sup>7,14,15,23</sup> In other words, the mode-by-mode evaluation of the Faddeev–Popov path integral in the one-loop approximation agrees with the result obtained, for the same path integral, by using the method of conformal variations with the corresponding Schwinger–DeWitt asymptotic expansion.<sup>7,11</sup> For example, in a portion of flat Euclidean four-space bounded by two concentric three-spheres, the contribution to  $\zeta(0)$  of the decoupled normal mode cancels exactly the one resulting from transverse modes. Moreover, on the four-dimensional disk, transverse modes give contributions of opposite signs to  $\zeta(0)$  with magnetic or electric boundary conditions,<sup>7</sup> whereas the  $\zeta(0)$  value is the same for both boundary conditions and equal to  $-31/90$ .<sup>14,15</sup> Thus, in problems where the background is four-dimensional and the bound-

ary is three-dimensional, the covariant  $\zeta$ -function calculations, including ghost modes, give results in disagreement with a calculation relying on boundary conditions for the fields and reduction to physical degrees of freedom.<sup>7,14,15,24</sup> Further evidence of the nontrivial role played by ghost modes in curved backgrounds had been obtained, much earlier, in Ref. 25. Hence, we find it nontrivial that the ghost formalism gives results in complete agreement with Boyer's investigation.<sup>2</sup> Note also that, in the case of perfectly conducting parallel plates, the ghost contribution cancels the one due to tangential components of the potential (see Sec. 4.5 of Ref. 7). This differs from the cancellations found in our paper in the presence of spherical symmetry.

The main open problem is now the explicit proof that the Casimir energy is independent of the choice of gauge-averaging functional. This task is as difficult as crucial to obtain a thorough understanding of the quantized electromagnetic field in the presence of bounding surfaces. In general, as shown in Sec. 4, one has then to study entangled eigenvalue equations for temporal, normal and longitudinal modes. The general solution is not (obviously) expressed in terms of well known special functions (cf., for example, Ref. 14). A satisfactory understanding of this problem is still lacking, while its solution would be of great relevance both for the foundations and for the applications of quantum field theory. In covariant gauges, coupled eigenvalue equations with arbitrary gauge parameter also lead to severe technical problems, which are described in Appendix B. A further set of nontrivial applications lies in the analysis of more involved geometries, where spherical symmetry no longer holds, or in the investigation of media which are not perfect conductors.<sup>26–28</sup> Thus, there exists increasing evidence that the study of Casimir energies will continue to play an important role in quantum field theory in the years to come, and that a formalism relying on potentials and ghost fields is, indeed, crucial on the way towards a better understanding of quantized fields.

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## APPENDIX A:

One can consider, in general, a  $(d + 1)$ -dimensional disk with metric

$$g = dr \otimes dr + r^2 \Omega_d, \quad r \in [0, R], \quad (\text{A1})$$

where  $\Omega_d$  is the metric on the  $d$ -dimensional sphere of unit radius. The action of the Laplace operator on a scalar field, say  $\phi$ , is found to be

$$\Delta \phi = \left( \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} {}^{(d)}\Delta \right) \phi, \quad (\text{A2})$$

whereas the action of the same operator on normal and tangential components of a one-form is

$$(\Delta A)_r = \left( \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} {}^{(d)}\Delta - \frac{d}{r^2} \right) A_r - \frac{2}{r^3} A_p{}^{|p}, \quad (\text{A3})$$

$$(\Delta A)_k = \left( \frac{\partial^2}{\partial r^2} + \frac{(d-2)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} {}^{(d)}\Delta - \frac{(d-1)}{r^2} \right) A_k + \frac{2}{r} \partial_k A_r. \quad (\text{A4})$$

With our notation,  ${}^{(d)}\Delta \equiv_{|p} {}^{|p}$  is the Laplacian on a  $d$ -sphere of unit radius. When  $d = 2$ , its action on spherical harmonics is such that

$${}^{(2)}\Delta Y_{lm}(\theta, \varphi) = -l(l+1)Y_{lm}(\theta, \varphi). \quad (\text{A5})$$

Another useful formula in the course of deriving Eq. (2.11) from Eq. (2.8) is the one according to which<sup>29</sup>

$${}^{(2)}\Delta \partial_k Y_{lm}(\theta, \varphi) = [-l(l+1) + 1] \partial_k Y_{lm}(\theta, \varphi). \quad (\text{A6})$$

## APPENDIX B:

A naturally occurring question is whether one can prove explicitly that the Casimir energy is independent of the value of the  $\alpha$  parameter in the operator (1.11). Indeed, from Eqs. (2.4), (2.7), (2.8), jointly with the form of the Lorenz gauge-averaging functional:

$$\Phi_L(A) = -\frac{\partial A_t}{\partial t} + \frac{\partial A_r}{\partial r} + \frac{1}{r^2} A_k^{[k]} + \frac{2}{r} A_r, \quad (\text{B1})$$

one finds that

$$-P_{t\nu}A^\nu = \left(\frac{\omega^2}{\alpha} + \Delta\right) A_t + i\omega \left(\frac{1}{\alpha} - 1\right) \left(\frac{\partial A_r}{\partial r} + \frac{1}{r^2} A_k^{[k]} + \frac{2}{r} A_r\right), \quad (\text{B2})$$

$$-P_{r\nu}A^\nu = (\omega^2 + \Delta) A_r + i\omega \left(1 - \frac{1}{\alpha}\right) \frac{\partial A_t}{\partial r} - \left(1 - \frac{1}{\alpha}\right) \frac{\partial}{\partial r} \left(\frac{\partial A_r}{\partial r} + \frac{1}{r^2} A_k^{[k]} + \frac{2}{r} A_r\right), \quad (\text{B3})$$

$$-P_{k\nu}A^\nu = (\omega^2 + \Delta) A_k + i\omega \left(1 - \frac{1}{\alpha}\right) \frac{\partial A_t}{\partial x^k} - \left(1 - \frac{1}{\alpha}\right) \frac{\partial}{\partial x^k} \left(\frac{\partial A_r}{\partial r} + \frac{1}{r^2} A_p^{[p]} + \frac{2}{r} A_r\right). \quad (\text{B4})$$

As in Sec. 2, we now perform the analytic continuation  $\omega \rightarrow iM$ . For arbitrary values of  $\alpha$ , this remains a well defined procedure, but it no longer leads to a system admitting Bessel type solutions. By contrast, by virtue of the equations (2.1)–(2.3) and (B.2)–(B.4), one finds a coupled system of three ordinary differential equations with variable coefficients. On defining the column vector

$$u \equiv \begin{pmatrix} a_l \\ b_l \\ c_l \end{pmatrix}, \quad (\text{B5})$$

jointly with the nine operators

$$A_{11} \equiv \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} + \left(1 - \frac{1}{\alpha}\right) M^2, \quad (\text{B6})$$

$$A_{12} \equiv M \left(1 - \frac{1}{\alpha}\right) \left(\frac{d}{dr} + \frac{2}{r}\right), \quad (\text{B7})$$

$$A_{13} \equiv M \left(\frac{1}{\alpha} - 1\right) \frac{l(l+1)}{r^2}, \quad (\text{B8})$$

$$A_{21} \equiv M \left( \frac{1}{\alpha} - 1 \right) \frac{d}{dr}, \quad (\text{B9})$$

$$A_{22} \equiv \frac{1}{\alpha} \frac{d^2}{dr^2} + \frac{2}{\alpha} \frac{1}{r} \frac{d}{dr} - \left( \frac{2}{\alpha} + l(l+1) \right) \frac{1}{r^2}, \quad (\text{B10})$$

$$A_{23} \equiv \left( 1 - \frac{1}{\alpha} \right) \frac{l(l+1)}{r^2} \frac{d}{dr} + \frac{2}{\alpha} \frac{l(l+1)}{r^3}, \quad (\text{B11})$$

$$A_{31} \equiv M \left( \frac{1}{\alpha} - 1 \right), \quad (\text{B12})$$

$$A_{32} \equiv \left( \frac{1}{\alpha} - 1 \right) \frac{d}{dr} + \frac{2}{\alpha} \frac{1}{r}, \quad (\text{B13})$$

$$A_{33} \equiv \frac{d^2}{dr^2} - \frac{l(l+1)}{\alpha r^2}, \quad (\text{B14})$$

the resulting system reads (with summation over the repeated index  $p = 1, 2, 3$ )

$$A_{jp} u_p = M^2 u_j, \quad (\text{B15})$$

for all  $j = 1, 2, 3$ . Remarkably, the temporal, normal and longitudinal modes are now *all coupled*, but the system of equations is so complicated that no explicit solution in terms of well known special functions can be obtained, as far as the authors can see. Nevertheless, if  $\alpha \neq 1$ , regular basis functions still exist, since  $r = 0$  is a Fuchsian singularity of the system (B.15). This implies that the desired solutions can be expressed, in the interior problem, in the form

$$r^\rho \sum_{k=0}^{\infty} \eta_{l,k} r^k,$$

where  $\rho$  is a real-valued parameter, and  $\eta_{l,k}$  are some coefficients. One should also say that, in the quantum cosmological backgrounds studied over the last few years,<sup>7</sup> the problem is less severe, since only normal and longitudinal modes remain coupled for arbitrary values of  $\alpha$  (but in that case there is no time variable, the background is four-dimensional and the boundary is three-dimensional). Only the transverse modes are decoupled, and obey the

equation (2.9). Moreover, the modes  $a_0$  and  $b_0$  are coupled as well (unlike the case when  $\alpha = 1$ ), and obey the following system of three eigenvalue equations:

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \left(1 - \frac{1}{\alpha}\right) M^2 \right] a_0 + M \left(1 - \frac{1}{\alpha}\right) \left( \frac{db_0}{dr} + \frac{2}{r} b_0 \right) = M^2 a_0, \quad (\text{B16})$$

$$\left[ \frac{1}{\alpha} \frac{d^2}{dr^2} + \frac{2}{\alpha r} \frac{d}{dr} - \frac{2}{\alpha r^2} \right] b_0 + M \left(\frac{1}{\alpha} - 1\right) \frac{da_0}{dr} = M^2 b_0, \quad (\text{B17})$$

$$M \left(\frac{1}{\alpha} - 1\right) a_0 + \left(\frac{1}{\alpha} - 1\right) \frac{db_0}{dr} + \frac{2}{\alpha} \frac{b_0}{r} = 0. \quad (\text{B18})$$

Similar technical problems occur on considering other choices of the gauge-averaging functional, e.g.

$$\Phi(A) \equiv \frac{\partial A_r}{\partial r} + \frac{1}{r^2} A_k^{|k} + \frac{2}{r} A_r,$$

and hence we only write down the eigenvalue equations (B.15) which are obtained in the Lorenz case.

Since in Sec. 4 we have found it quite helpful to perform a comparison with the analysis of a flat boundary, here we describe the same procedure for the Lorenz gauge. On using again the plane-wave expansion (4.16) for the electromagnetic potential, one gets the following compatibility condition for field equations:

$$F \equiv \det \begin{pmatrix} \omega^2 - k^2 - \beta\omega^2 & \beta\omega k_x & \beta\omega k_y & \beta\omega k_z \\ \beta\omega k_x & k^2 - \omega^2 - \beta k_x^2 & -\beta k_x k_y & -\beta k_x k_z \\ \beta\omega k_y & -\beta k_x k_y & k^2 - \omega^2 - \beta k_y^2 & -\beta k_y k_z \\ \beta\omega k_z & -\beta k_x k_z & -\beta k_y k_z & k^2 - \omega^2 - \beta k_z^2 \end{pmatrix} = 0, \quad (\text{B19})$$

where  $\beta \equiv 1 - \frac{1}{\alpha}$ . The direct calculation shows that

$$F = -\frac{1}{\alpha} (\omega^2 - k^2)^4. \quad (\text{B20})$$

Hence one can see that the dispersion relation does not depend on  $\alpha$ . The same holds for ghost modes, where the similar condition reads

$$(\omega^2 - k^2)^2 = 0. \quad (\text{B21})$$

Thus, we have obtained again the correct dispersion relation, with the correct power of  $(\omega^2 - k^2)$ .

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